ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 324 (2006) 1–13
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Polynomials without repelling periodic point of given period ☆

Jianming Chang

Department of Mathematics, Changshu Institute of Technology, Changshu, Jiangsu 215500, PR China

Received 7 August 2005

Available online 28 December 2005

Submitted by H.W. Broer

Abstract

Bergweiler proved that for any given integer $k \geqslant 2$, every polynomial P of degree $d \geqslant 2$ has at least one repelling periodic cycle of period k unless $(k,d) \in \{(2,2),(2,3),(2,4),(3,2)\}$. Here we classified these exceptional polynomials. We also showed that the Julia sets of these exceptional polynomials are connected.

Keywords: Polynomial; Iterate; Fixed point; Periodic point and cycle; Repelling periodic point and cycle; Nonrepelling periodic point and cycle

1. Introduction and main results

Let $P(z)$ be a polynomial. Then the kth iterates of P, denoted by $P^k(z)$, are defined inductively by $P^1(z) = P(z)$ and $P^k(z) = P^{k-1} \circ P(z)$ for $k \geqslant 2$.

Let $z_0 \in \mathbb{C}$. If there exists a positive integer $p \in \mathbb{N}$ such that $P^p(z_0) = z_0$ but $P^j(z_0) \neq z_0$ for any $1 \leqslant j \leqslant p-1$, then z_0 is said to be a periodic point of period p of P, and the corresponding cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ is said to be a periodic cycle of period p. A periodic point of period 1 is said to be a fixed point. Define the multiplier of the periodic point z_0 (and the corresponding cycle) by $\lambda = (P^p)'(z_0)$. According to $|\lambda| < 1$ $(\lambda = 0)$, $|\lambda| = 1$, or $|\lambda| > 1$, the periodic point z_0 (and the corresponding cycle) is said to be attracting (superattracting), neutral,

☆ Supported by the NNSF of China (Grant No. 10471065), and the NSF of Education Department of Jiangsu Province (Grant No. 04KJD110001).

E-mail addresses: jmwchang@pub.sz.jsinfo.net, jmchang@cslg.edu.cn.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.11.075

or repelling. If $|\lambda| = 1$, then according to whether there is some integer m such that $(\lambda)^m = 1$ or not, z_0 is said to be rationally neutral or irrationally neutral. An important fact is that ∞ is a superattracting fixed point (see [4]).

Fixed points and periodic points play an important part in complex dynamics. For example, the Julia set of a polynomial is the closure of the set of all its repelling periodic points (see [4]).

Baker is the first who studied the existence of period points of given period for polynomials and rational functions. For polynomials, he proved

Theorem A. [1, Theorem 2] *Let $P(z)$ be a polynomial of degree $d \geqslant 2$ and $k \geqslant 2$ be an integer. Then $P(z)$ has at least one periodic cycle of period k unless $(k, d) = (2, 2)$ and $P(z)$ is similar to $z^2 - 3/4$.*

Here and in the sequel, we say two polynomials P and Q are similar if there exist constants $a(\neq 0), b$ such that $Q(az + b) = aP(z) + b$. See [1].

Later, Bergweiler considered the analogue for repelling period points and proved

Theorem B. [2, §1.4, Satz 1] *Let $P(z)$ be a polynomial of degree $d \geqslant 2$ and $k \geqslant 2$ be an integer. Then $P(z)$ has at least one repelling periodic cycle of period k unless $(k, d) \in \{(2, 2), (2, 3), (2, 4), (3, 2)\}$.*

He also gave examples to show that all exceptional cases can occur. The aim of this note is to classify these exceptional polynomials. We prove

Theorem 1. *Let P be a polynomial of degree $d \geqslant 2$, and $k \geqslant 2$ be an integer such that $(k, d) \in \{(2, 2), (2, 3), (2, 4), (3, 2)\}$. If P has no repelling periodic cycle of period k, then P is similar to one of the following polynomials: for $k = 2$,*

(a) $Q(z) = z^2 + c$, *for* $c \in \{z\colon |z + 1| \leqslant 1/4\}$;
(b) $Q(z) = z^3 - (2 + 3c^2)z + 2c^3$, *for* $c \in \{z\colon |z^2 - 1/36| \leqslant 1/36\}$; *and*
(c) $Q(z) = z^4 + (-1 \pm 2i)z$, *while for* $k = 3$;
(d) $Q(z) = z^2 - 7/4$.

The exceptional polynomials in (a) and (d) are of degree 2, so where these polynomials are located relative the Mandelbrot set is worth researching. Here the Mandelbrot set, see [4, p. 124], is defined by

$$\begin{aligned}\mathcal{M} &= \left\{c \in \mathbb{C}\colon \text{ the orbit } \left\{P_c^n(0)\colon\, n \in \mathbb{N}\right\} \text{ of } P_c = z^2 + c \text{ at } 0 \text{ is bounded}\right\}\\ &= \left\{c \in \mathbb{C}\colon \text{ the Julia set } J(P_c) \text{ of } P_c = z^2 + c \text{ is connected}\right\}.\end{aligned}$$

With calling $P_c = z^2 + c$ for $c \in \mathcal{M}$ a Mandelbrot polynomial, we show that all exceptional polynomials in (a) and (d) are Mandelbrot polynomials, and thus have connected Julia sets.

Indeed, we have

Theorem 2. *The Julia sets of all polynomials in* (a), (b), (c) *and* (d) *are connected.*

2. Lemmas

Let P be polynomial of degree $d \geqslant 2$. Let z_0 be a rationally neutral periodic point of period p of P, then the multiplier $(P^p)'(z_0)$ must be a primitive mth root of one for some positive integer m. Thus near z_0, we have

$$P^{pm}(z) = z + \alpha(z - z_0)^{km+1} + \cdots \tag{1}$$

(see [4, p. 41]) for some constant $\alpha \neq 0$ and some positive integer k. Now we define the multiplicity and dynamical multiplicity of the cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ to be $\nu = km + 1$ and k, respectively.

If z_0 is a periodic point of period p of P which is not rationally neutral, then define the multiplicity and the dynamical multiplicity of the cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ both to be one (see [1,5]).

Then we have

Lemma 1. ([5, Theorem 3], cf. [4, p. 100, Theorem 1.2]) *Let P be a polynomial of degree $d \geqslant 2$. Then P has at most $d - 1$ nonrepelling periodic cycles in $\mathbb{C}$ counting dynamical multiplicity.*

Lemma 2. (See [1,5].) *Let P be polynomial of degree $\geqslant 2$ and let $k \geqslant 2$ be an integer. Suppose that z_0 is a fixed point of P^k in $\mathbb{C}$. Then*

(1) *z_0 is a periodic point of period p of P which is a factor of k.*
(2) *All points in the cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ are roots of the equation $P^k(z) - z = 0$ with the same multiplicity μ. And if $\mu > 1$, then there is an integer α such that $p\alpha (\leqslant k)$ is a factor of k and $\mu = \alpha N + 1$, where N is the dynamical multiplicity of the cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$.*
(3) *Let ν be the multiplicity of the cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$. Then*
   (i) *for rationally neutral, $\nu = \mu$ for $\mu > 1$, and $\nu > 1$ for $\mu = 1$;*
   (ii) *for not rationally neutral, $\nu = \mu = 1$.*

To prove Theorem 2, we need some other results. Let $P(z)$ be a polynomial of degree $d \geqslant 2$. Then the point ∞ and the points $\zeta \in \mathbb{C}$ at which $P'(\zeta) = 0$ are called the critical points of P.

Let z_0 be an attracting fixed point of P. Then the set consisting of z such that $P^n(z) \to z_0$ is called the basin of attracting of z_0 with respect to P, and its component that contains z_0 is called the immediate basin of attracting of z_0 with respect to P.

Now let $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ be an attracting periodic cycle of period p of P. Then each $P^j(z_0)$ is an attracting fixed point of P^p. Let U_j be the immediate basin of attracting of $P^j(z_0)$ with respect to P^p. Then the union $\bigcup_{j=0}^{p-1} U_j$ is called an immediate basin of attraction associated to the attracting periodic cycle $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\}$ (see [4, p. 58]).

Lemma 3. [4, p. 59, Theorem 2.2] *The immediate basin of attraction associated to an attracting periodic cycle contains at least one critical point.*

Now let $\{z_0, P(z_0), \ldots, P^{p-1}(z_0)\} (\subset \mathbb{C})$ be a rationally neutral cycle of period p. Then there exists a smallest integer $m \geqslant 1$ such that $[(P^p)'(z_0)]^m = 1$, so that near z_0, we have (1). Thus by the well-known Leau–Fatou's flower theorem, for each $0 \leqslant j \leqslant p - 1$, the Fatou set of P has km components $U_{j,i}$ $(1 \leqslant i \leqslant km)$, called Leau domains, such that $P^j(z_0) \in \partial U_{j,i}$, and in $U_{j,i}$, $P^{np}(z) \to P^j(z_0)$ as $n \to \infty$. Furthermore, these pkm Leau domains can be divided into k groups, each group G has pm Leau domains such that $P(G) = G$. That is, each group G can be written as $G = \{P^j(U),\ 0 \leqslant j \leqslant pm - 1\}$ $(P^{pm}(U) = U)$. The group G is called a cycle of Leau domains associated to the rationally neutral cycle $\{z_0, P(z_0), \dots, P^{p-1}(z_0)\}$. The union $\bigcup_{j=0}^{pm-1} P^j(U)$ is called an immediate basin of attraction associated to the rationally neutral cycle $\{z_0, P(z_0), \dots, P^{p-1}(z_0)\}$. Thus there are k immediate basins of attraction associated to the rationally neutral cycle $\{z_0, P(z_0), \dots, P^{p-1}(z_0)\}$ (see [4, p. 60]).

Lemma 4. [4, p. 60, Theorem 2.3] *Each immediate basin of attraction associated to a rationally neutral periodic cycle contains at least one critical point.*

Lemma 5. [4, p. 66, Theorem 4.1] *For a polynomial P of degree $\geqslant 2$, its Julia set $J(P)$ is connected if and only if there is no finite critical point of P in the immediate basin $A(\infty)$ of attraction associated to the superattracting fixed point ∞.*

Lemma 6. [4, p. 124, Theorem 1.2] *The Mandelbrot set $\mathcal{M}$ is a closed simply connected subset of the disk* $\{c\colon |c| \leqslant 2\}$, *which meets the real axis in the interval* $[-2, 1/4]$.

The next result is an analogue of Lemma 6 for general polynomials of degree $d \geqslant 2$. Set

$$\mathcal{C}_d = \bigl\{\mathbf{a} = (a_{d-2}, \dots, a_0) \in \mathbb{C}^{d-1}\colon \text{ the Julia set } J(P_{\mathbf{a}}) \text{ of } P_{\mathbf{a}} = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \text{ is connected}\bigr\}.$$

Then $\mathcal{C}_2 = \mathcal{M}$. By [3], we have

Lemma 7. (See [3, Chapter I].) *The set $\mathcal{C}_d \subset \mathbb{C}^{d-1}$ is closed and simply connected.*

3. Proof of Theorem B

Here we give a proof of Theorem B for the sake of readers.

Proof of Theorem B. Suppose that P has no repelling periodic cycle of period k.

Let $z_0 \in \mathbb{C}$ be a fixed point of P^k. Then we get a periodic cycle

$$\bigl\{P^0(z_0)(= z_0), P^1(z_0), \dots, P^{j-1}(z_0)\bigr\}$$

of period j. Obviously, j is a factor of k.

Let j be a factor of k. Assume that P has $n_j (\geqslant 0)$ cycles

$$\Gamma_{j,i} = \bigl\{P^0(z_{j,i}), P^1(z_{j,i}), \dots, P^{j-1}(z_{j,i})\bigr\}, \quad i = 1, 2, \dots, n_j, \tag{2}$$

of period j. And assume that in these cycles, there are $(0 \leqslant) m_j (\leqslant n_j)$ nonrepelling cycles $\Gamma_{j,i}$ $(1 \leqslant i \leqslant m_j)$. By assumption, we have

$$n_k = m_k. \tag{3}$$

Define

$$\nu^*_{j,i} = \nu_{j,i} \quad \text{for } \mu_{j,i} > 1, \quad \text{and} \quad \nu^*_{j,i} = 1 \quad \text{for } \mu_{j,i} = 1, \tag{4}$$

where $\mu_{j,i}$ is the multiplicity of $z_{j,i}$ as a root of the equation $P^k(z) - z = 0$, and $\nu_{j,i}$ is the multiplicity of the cycle $\Gamma_{j,i}$ starting at $z_{j,i}$.

By Lemma 2, we have

$$\nu^*_{j,i} = \mu_{j,i}. \tag{5}$$

Thus we have

$$P^k(z) = z + C_k \prod_{j|k} \left\{ \prod_{i=1}^{m_j} \left[\prod_{\zeta\in\Gamma_{j,i}} (z-\zeta) \right]^{\nu^*_{j,i}} \prod_{i=m_j+1}^{n_j} \left[\prod_{\zeta\in\Gamma_{j,i}} (z-\zeta) \right] \right\}, \tag{6}$$

where C_k is a nonzero constant. It follows that

$$d^k = \deg\bigl(P^k\bigr) = \sum_{j|k} j \left(\sum_{i=1}^{m_j} \nu^*_{j,i} + n_j - m_j \right). \tag{7}$$

Now denote by $N_{j,i}$ the dynamical multiplicity of the cycle $\Gamma_{j,i}$. Then by Lemma 1, we have

$$\sum_{j|k} m_j \leqslant \sum_{j|k} \sum_{i=1}^{m_j} N_{j,i} \leqslant d-1. \tag{8}$$

And as in [5, p. 144], by Lemma 2, there exist nonnegative integers $\alpha_{j,i}\,(\leqslant k)$ such that

$$\nu^*_{j,i} = \alpha_{j,i} N_{j,i} + 1 \tag{9}$$

and

$$j\alpha_{j,i} \leqslant k. \tag{10}$$

Thus by (3), (7), (9) and (10), we get

$$\begin{aligned} d^k &= \sum_{j|k} j \left(\sum_{i=1}^{m_j} \alpha_{j,i} N_{j,i} + n_j \right) = \sum_{j|k} \sum_{i=1}^{m_j} (j\alpha_{j,i}) N_{j,i} + \sum_{j|k} j n_j \\ &\leqslant k \sum_{j|k} \sum_{i=1}^{m_j} N_{j,i} + \sum_{j|k,\ j<k} j n_j + k m_k \leqslant 2k \sum_{j|k} \sum_{i=1}^{m_j} N_{j,i} + \sum_{j|k,\ j<k} j n_j. \end{aligned} \tag{11}$$

Note that $1 \leqslant n_1 \leqslant d$ and $n_1 + jn_j \leqslant \deg(P^j) = d^j$ for $j > 1$. Since for $j < k$ and $j \mid k$, we have $j \leqslant [k/2]$, where $[k/2]$ is the largest integer not exceeding $k/2$. Thus, by (8) and (11), we see that

$$\begin{aligned} d^k &\leqslant 2k(d-1) + d + \sum_{j=2}^{[k/2]} \bigl(d^j - 1\bigr) = 2k(d-1) + \frac{d(d^{[k/2]}-1)}{d-1} - \bigl([k/2]-1\bigr) \\ &\leqslant 2d^{k/2} + 2kd - \frac{5k+1}{2} \leqslant 2(k+1)d^{k/2} - \frac{5k+1}{2}. \end{aligned} \tag{12}$$

It follows that $d^{k/2} < 2k+1$ so that $d < (2k+1)^{2/k}$. Let

$$f(x) = (2x+1)^{2/x}.$$

Note that the function $f(x)$ is decreasing for $x > 1$, as for $x > 1$

$$f'(x) = -\frac{2}{x^2}(2x+1)^{2/x} \left(\log(2x+1) - \frac{2x}{2x+1} \right) < 0.$$

Thus we get for $k \geqslant 9$, $d < f(9) < 2$. This contradicts $d \geqslant 2$. Hence $k \leqslant 8$. Again by $d < f(k)$, for $k \geqslant 4$, $d < f(4) = 3$, and for $k = 3$, $d < f(3) < 4$, and for $k = 2$, $d < f(2) = 5$.

However for $(k,d) = (8,2)$, $(7,2)$, $(6,2)$, $(5,2)$, $(4,2)$ and $(3,3)$, by (12), we get the following contradictions respectively: $256 \leqslant 87/2$, $128 \leqslant 16\sqrt{2} + 10$, $64 \leqslant 49/2$, $32 \leqslant 8\sqrt{2} + 7$, $16 \leqslant 27/2$ and $27 \leqslant 6\sqrt{3} + 10$.

Thus we see that $(k,d) \in \{(3,2),(2,2),(2,3),(2,4)\}$.

Theorem B is proved. $\square$

4. Proof of Theorem 1

Suppose that P has no repelling periodic cycle of period k. Next we consider four cases.

Case 1. Assume $k = 3$. Then $d = 2$.

Then by (8) and (11), we have

$$\begin{aligned} 8 = d^3 &= n_1 + \sum_{i=1}^{m_1} \alpha_{1,i} N_{1,i} + \sum_{i=1}^{m_3} (3\alpha_{3,i}) N_{3,i} + 3m_3 \\ &\leqslant n_1 + 3\left(\sum_{i=1}^{m_1} N_{1,i} + \sum_{i=1}^{m_3} N_{3,i}\right) + 3m_3 \leqslant d + 6\left(\sum_{i=1}^{m_1} N_{1,i} + \sum_{i=1}^{m_3} N_{3,i}\right) \\ &\leqslant d + 6(d-1) = 8. \end{aligned} \tag{13}$$

Thus we see that $n_1 = 2$, $m_1 = 0$ and $m_3 = \sum_{i=1}^{m_3} N_{3,i} = 1$ and $\alpha_{3,i} = 1$. Hence P and P^3 are of the form:

$$P(z) = z + c(z-\alpha)(z-\beta), \tag{14}$$
$$P^3(z) = z + c^7(z-\alpha)(z-\beta)\big[(z-z_0)\big(z-P(z_0)\big)\big(z-P^2(z_0)\big)\big]^2, \tag{15}$$

where c is a nonzero constant, α, β are two repelling fixed points of P, and $\{z_0, P(z_0), P^2(z_0)\}$ is a rationally neutral cycle of period 3 of P.

Let

$$Q(z) = \phi^{-1} \circ P \circ \phi(z), \tag{16}$$

where $\phi(z) = z/c + (\beta+\alpha)/2$. Then we see that

$$Q(z) = z + z^2 + \delta, \tag{17}$$
$$Q^3(z) = z + \big(z^2+\delta\big)\big[\big(z-z_0'\big)\big(z-Q\big(z_0'\big)\big)\big(z-Q^2\big(z_0'\big)\big)\big]^2, \tag{18}$$

where $\delta = -(\alpha-\beta)^2c^2/4$ is a nonzero constant.

By (17) with some computation, we have

$$\begin{aligned} R(z) &= \frac{Q^3(z)-z}{Q(z)-z} \\ &= z^6 + 4z^5 + (3\delta+8)z^4 + (8\delta+10)z^3 + \big(3\delta^2+12\delta+9\big)z^2 + \big(4\delta^2+10\delta+6\big)z \\ &\quad + \delta^3 + 4\delta^2 + 5\delta + 3. \end{aligned} \tag{19}$$

On the other hand, by (17) and (18), we have R has the form

$$\begin{aligned}R(z) &= \left(z^3 + Az^2 + Bz + C\right)^2 \\ &= z^6 + 2Az^5 + \left(A^2 + 2B\right)z^4 + (2AB + 2C)z^3 + \left(2AC + B^2\right)z^2 \\ &\quad + 2BCz + C^2,\end{aligned} \tag{20}$$

where A, B, C are constants. Comparing the coefficients of (19) and (20), we get

$$\begin{gathered}2A = 4, \qquad A^2 + 2B = 3\delta + 8, \qquad 2AB + 2C = 8\delta + 10, \\ 2AC + B^2 = 3\delta^2 + 12\delta + 9, \qquad 2BC = 4\delta^2 + 10\delta + 6, \qquad C^2 = \delta^3 + 4\delta^2 + 5\delta + 3.\end{gathered}$$

It follows that $A = 2$, $B = -1$, $C = -1$ and $\delta = -2$.[1] Thus we have

$$Q(z) = z + z^2 - 2 \quad \text{and} \quad Q^3(z) = z + \left(z^2 - 2\right)\left(z^3 + 2z^2 - z - 1\right)^2. \tag{21}$$

Let $T(z) = \psi^{-1} \circ Q \circ \psi(z)$, where $\psi(z) = z - 1/2$. Then we see that

$$T(z) = z^2 - \frac{7}{4} \quad \text{and} \quad T^3(z) = z + \left[T(z) - z\right]\left(z^3 + \frac{1}{2}z^2 - \frac{9}{4}z - \frac{1}{8}\right)^2.$$

Thus P is similar to $z^2 - 7/4$.

Case 2. Assume $k = 2$. Then $2 \leqslant d \leqslant 4$.

Then by (8) and (11), we have

$$\begin{aligned}d^2 &= n_1 + \sum_{i=1}^{m_1} \alpha_{1,i} N_{1,i} + \sum_{i=1}^{m_2} (2\alpha_{2,i}) N_{2,i} + 2m_2 \leqslant d + 2\left(\sum_{i=1}^{m_1} N_{1,i} + \sum_{i=1}^{m_2} N_{2,i}\right) + 2m_2 \\ &\leqslant d + 4\left(\sum_{i=1}^{m_1} N_{1,i} + \sum_{i=1}^{m_2} N_{2,i}\right) \leqslant d + 4(d-1) = 5d - 4.\end{aligned} \tag{22}$$

Case 2.1. Assume $d = 4$. Then by (22), we see that $n_1 = 4$, $m_1 = 0$, $m_2 = \sum_{i=1}^{m_2} N_{2,i} = 3$ and $\alpha_{2,i} = 1$. Thus as in Case 1, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z) = z + z^4 + \delta z^2 + \sigma z + \tau, \tag{23}$$

$$Q^2(z) = z + \left(z^4 + \delta z^2 + \sigma z + \tau\right) \prod_{j=1}^{3} \left[\left(z - z'_j\right)\left(z - Q\left(z'_j\right)\right)\right]^2, \tag{24}$$

where δ, σ, τ are constants. Next as in Case 1, we first compute out

$$\begin{aligned}R(z) &= \frac{Q^2(z) - z}{Q(z) - z} \\ &= z^{12} + 3\delta z^{10} + (4 + 3\sigma)z^9 + \left(3\tau + 3\delta^2\right)z^8 + (6\delta\sigma + 8\delta)z^7 \\ &\quad + \left(6 + \delta^3 + 6\delta\tau + 3\sigma^2 + 8\sigma\right)z^6 + \left(3\delta^2\sigma + 4\delta^2 + 6\sigma\tau + 8\tau\right)z^5\end{aligned}$$

[1] This process can be done with the aid of Maple. The Maple command is > solve (identity $(R = H, z)$, $\{\delta, A, B, C\}$), where R and H represent the polynomials (19) and (20), respectively.

$$\begin{aligned}&+\big(7\delta+3\delta\sigma^2+3\tau^2+3\delta^2\tau+8\delta\sigma\big)z^4+\big(6\delta\sigma\tau+8\delta\tau+4\sigma^2+\sigma^3+6\sigma+4\big)z^3\\&+\big(8\sigma\tau+\delta^2+3\sigma^2\tau+6\tau+3\delta\tau^2\big)z^2\\&+\big(2\delta+4\tau^2+3\sigma\tau^2+\delta\sigma\big)z+2+\sigma+\tau^3+\delta\tau.\end{aligned}\tag{25}$$

On the other hand, by (23) and (24), we have

$$\begin{aligned}R(z)&=\big(z^6+Az^5+Bz^4+Cz^3+Mz^2+Nz+S\big)^2\\&=z^{12}+2Az^{11}+\big(A^2+2B\big)z^{10}+(2AB+2C)z^9+\big(B^2+2AC+2M\big)z^8\\&\quad+(2BC+2N+2AM)z^7+\big(C^2+2S+2BM+2AN\big)z^6\\&\quad+(2BN+2CM+2AS)z^5+\big(2BS+2CN+M^2\big)z^4\\&\quad+(2CS+2MN)z^3+\big(N^2+2MS\big)z^2+2NSz+S^2,\end{aligned}\tag{26}$$

where A, B, C, M, N, S are constants. Then comparing the coefficients of (25) and (26) shows that $\delta=0$, $\sigma=-2\pm 2i$, $\tau=0$, $A=0$, $B=0$, $C=-1\pm 3i$, $M=0$, $N=0$, $S=-1\mp i$. Thus we get

$$\begin{aligned}&Q(z)=z+z^4+(-2\pm 2i)z=z^4+(-1\pm 2i)z,\\&Q^2(z)=z+\big[z^4+(-2\pm 2i)z\big]\big[z^6+(-1\pm 3i)z^3-1\mp i\big]^2.\end{aligned}$$

Case 2.2. Assume $d=3$. Then by (22), we see that $\sum_{i=1}^{m_1}N_{1,i}+\sum_{i=1}^{m_2}N_{2,i}=2$ and $m_2\geqslant 1$. Thus we have either $\sum_{i=1}^{m_1}N_{1,i}=\sum_{i=1}^{m_2}N_{2,i}=1$ or $\sum_{i=1}^{m_1}N_{1,i}=0$ and $\sum_{i=1}^{m_2}N_{2,i}=2$.

Case 2.2.1 *($\sum_{i=1}^{m_1}N_{1,i}=\sum_{i=1}^{m_2}N_{2,i}=1$).* Then we can see that $m_1=m_2=1$, $N_{1,1}=N_{2,1}=1$, and $n_1=3$, $\alpha_{1,1}=2$, $\alpha_{2,1}=1$, so that $\nu_{1,1}^*=3$ and $\nu_{2,1}^*=2$. Then as in Case 1, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z)=z+\big(z^2+\delta\big)(z+\mu),\tag{27}$$

$$Q^2(z)=z+\big(z^2+\delta\big)(z+\mu)^3\big[\big(z-z_0'\big)\big(z-Q\big(z_0'\big)\big)\big]^2.\tag{28}$$

By (28), we see that $[Q'(-\mu)]^2=(Q^2)'(-\mu)=1$. Since $Q'(-\mu)\neq 1$, we get $Q'(-\mu)=-1$. Thus by (27), we see that $1+\mu^2+\delta=-1$, so that $\delta=-\mu^2-2$. Thus by (27) and (28), we have

$$Q(z)=z+\big(z^2-\mu^2-2\big)(z+\mu),\tag{29}$$

$$Q^2(z)=z+\big(z^2-\mu^2-2\big)(z+\mu)^3\big[\big(z-z_0'\big)\big(z-Q\big(z_0'\big)\big)\big]^2.\tag{30}$$

By (29) with some computation, we have $Q^2(z)=z+(z^2-\mu^2-2)(z+\mu)^3H(z)$, where

$$H(z)=z^4-\big(2\mu^2+1\big)z^2-2\mu z+\mu^4+3\mu^2+1.\tag{31}$$

On the other hand, by (30), we have

$$H(z)=\big(z^2+Az+B\big)^2\tag{32}$$

for some constants A and B. Comparing the coefficients of (31) and (32) shows that this subcase cannot occur.

Case 2.2.2 *($\sum_{i=1}^{m_1}N_{1,i}=0$ and $\sum_{i=1}^{m_2}N_{2,i}=2$).* Then we have $m_1=0$, $n_1=d=3$ and $1\leqslant m_2\leqslant 2$. Thus in this subcase, P has three repelling fixed points.

Case 2.2.2.1 *($m_2 = 1$).* Then, as in Case 1, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z) = z + z^3 + \delta z + \sigma, \tag{33}$$

$$Q^2(z) = z + \left(z^3 + \delta z + \sigma\right)\left[\left(z - z_0'\right)\left(z - Q\left(z_0'\right)\right)\right]^3. \tag{34}$$

By (33) with some computation, we have

$$\begin{aligned} R(z) &= \frac{Q^2(z) - z}{Q(z) - z} \\ &= z^6 + (2\delta + 3)z^4 + 2\sigma z^3 + \left(\delta^2 + 3\delta + 3\right)z^2 + (3\sigma + 2\delta\sigma)z + \delta + 2 + \sigma^2. \end{aligned} \tag{35}$$

On the other hand, by (33) and (34), we have

$$\begin{aligned} R(z) &= \left(z^2 + Az + B\right)^3 \\ &= z^6 + 3Az^5 + \left(3B + 3A^2\right)z^4 + \left(A^3 + 6AB\right)z^3 + \left(3A^2B + 3B^2\right)z^2 \\ &\quad + 3AB^2z + B^3. \end{aligned} \tag{36}$$

Comparing the coefficients of (35) and (36) shows that $\delta = -3$, $\sigma = 0$, and then

$$Q(z) = z + z^3 - 3z = z^3 - 2z \quad \text{and} \quad Q^2(z) = z + \left(z^3 - 3z\right)\left(z^2 - 1\right)^3. \tag{37}$$

Case 2.2.2.2 *($m_2 = 2$).* Then, as in Case 1, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z) = z + z^3 + \delta z + \sigma, \tag{38}$$

$$Q^2(z) = z + \left(z^3 + \delta z + \sigma\right)\left[\left(z - z_0'\right)\left(z - Q\left(z_0'\right)\right)\right]^2\left(z - z_1'\right)\left(z - Q\left(z_1'\right)\right). \tag{39}$$

Then we again have (35). But by (38) and (39), we have

$$\begin{aligned} R(z) &= \left(z^2 + Az + B\right)^2\left(z^2 + Cz + D\right) \\ &= z^6 + (C + 2A)z^5 + \left(A^2 + 2B + 2AC + D\right)z^4 + \left(2AD + A^2C + 2BC + 2AB\right)z^3 \\ &\quad + \left(B^2 + 2ABC + A^2D + 2BD\right)z^2 + \left(2ABD + B^2C\right)z + B^2D \end{aligned} \tag{40}$$

for some constants A, B, C, D. Now comparing the coefficients of (35) and (40) shows that $\delta = -3 - 3A^2$, $\sigma = 2A^3$, where $A \neq 0$. Thus we have

$$Q(z) = z + z^3 - \left(3 + 3A^2\right)z + 2A^3 = z^3 - \left(2 + 3A^2\right)z + 2A^3, \tag{41}$$

$$Q^2(z) = z + \left[z^3 - \left(3 + 3A^2\right)z + 2A^3\right]\left(z^2 + Az - 1 - 2A^2\right)^2(z - A - 1)(z - A + 1). \tag{42}$$

By (41) and (42), $\{A + 1, A - 1\}$ is a periodic cycle of period 2, and its multiplier is

$$\left(Q^2\right)'(A + 1) = Q'(A - 1)Q'(A + 1) = 1 - 36A^2. \tag{43}$$

Since Q has no repelling cycle of period 2, by (43), we see that $|1 - 36A^2| \leqslant 1$. We claim if $|1 - 36A^2| \leqslant 1$, then Q has three repelling fixed points. Suppose Q has a nonrepelling fixed point z_0. Then by (41), we have

$$z_0^3-\left(3+3A^2\right)z_0+2A^3=0, \tag{44}$$
$$\lambda=Q'(z_0)=3z_0^2-2-3A^2, \tag{45}$$

and $|\lambda|\leqslant 1$. By (45), we get $3z_0^2=\lambda+2+3A^2$, and hence by (44),

$$\begin{aligned}0&=3\left[z_0^3-\left(3+3A^2\right)z_0+2A^3\right]=z_0\left(\lambda+2+3A^2\right)-3\left(3+3A^2\right)z_0+6A^3\\&=z_0\left(\lambda-7-6A^2\right)+6A^3.\end{aligned}$$

With $3z_0^2=\lambda+2+3A^2$, it follows that

$$36A^6=z_0^2\left(\lambda-7-6A^2\right)^2=\frac{1}{3}\left(\lambda+2+3A^2\right)\left(\lambda-7-6A^2\right)^2.$$

Thus we get

$$\lambda^3-\left(12+9A^2\right)\lambda^2+\left(21+18A^2\right)\lambda+98+315A^2+324A^4=0,$$

so that

$$\begin{aligned}&4\lambda^3-\left[49+\left(36A^2-1\right)\right]\lambda^2+\left[86+2\left(36A^2-1\right)\right]\lambda+\left(36A^2-1\right)^2+37\left(36A^2-1\right)\\&\quad+428=0.\end{aligned}$$

It follows that

$$\begin{aligned}428&\leqslant 4|\lambda|^3+\left(49+\left|36A^2-1\right|\right)|\lambda|^2+\left(86+2\left|36A^2-1\right|\right)|\lambda|+\left|36A^2-1\right|^2\\&\quad+37\left|36A^2-1\right|\\&\leqslant 4+50+88+1+37=180,\end{aligned}$$

which is a contradiction. Thus

$$A\in\left\{z\colon\ \left|z^2-\frac{1}{36}\right|\leqslant\frac{1}{36},\ z\neq 0\right\}. \tag{46}$$

Case 2.3. Assume $d=2$. Then by (22), we see that $\sum_{i=1}^{m_1}N_{1,i}+\sum_{i=1}^{m_2}N_{2,i}=d-1=1$. Thus we have either $\sum_{i=1}^{m_1}N_{1,i}=1$ and $\sum_{i=1}^{m_2}N_{2,i}=0$, or $\sum_{i=1}^{m_1}N_{1,i}=0$ and $\sum_{i=1}^{m_2}N_{2,i}=1$.

Case 2.3.1 *(*$\sum_{i=1}^{m_1}N_{1,i}=1$ *and* $\sum_{i=1}^{m_2}N_{2,i}=0$*)*. Then we see that $m_1=1$, $m_2=0$, $N_{1,1}=1$ and $n_1=\alpha_{1,1}=2$. In this case, as above, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z)=z+z^2-\delta^2=z+(z-\delta)(z+\delta), \tag{47}$$
$$Q^2(z)=z+(z-\delta)(z+\delta)^3. \tag{48}$$

By (47) with some computation, we have

$$Q^2(z)=z+(z-\delta)(z+\delta)\left(z^2+2z+2-\delta^2\right). \tag{49}$$

By (48) and (49), we see that $z^2+2z+2-\delta^2=(z+\delta)^2$. It follows that $\delta=1$. Thus

$$Q(z)=z+z^2-1\quad\text{and}\quad Q^2(z)=z+(z-1)(z+1)^3.$$

Let $T(z)=\psi^{-1}\circ Q\circ\psi(z)$, where $\psi(z)=z-1/2$. Then we see that

$$T(z) = z^2 - \frac{3}{4}. \tag{50}$$

Thus P is similar to $z^2 - 3/4$. It is the exceptional polynomial in Baker's Theorem A.

Case 2.3.2 *($\sum_{i=1}^{m_1} N_{1,i} = 0$ and $\sum_{i=1}^{m_2} N_{2,i} = 1$).* Then we see that $m_1 = 0,\ m_2 = 1,\ N_{2,1} = 1$ and $n_1 = d = 2$, $\alpha_{2,1} = 0$. In this case, as above, we can see that P is similar to a polynomial Q such that Q and iterate Q^2 are of the form:

$$Q(z) = z + z^2 - 1 - \eta^2, \tag{51}$$

$$Q^2(z) = z + \left(z^2 - 1 - \eta^2\right)(z - z_0)\left(z - Q(z_0)\right), \tag{52}$$

where η is a nonzero constant. By (51) and some computation, we have

$$Q^2(z) = z + \left(z^2 - 1 - \eta^2\right)(z - \eta + 1)(z + \eta + 1). \tag{53}$$

Thus $\{\eta - 1, -\eta - 1\} = \{z_0, Q(z_0)\}$ is a nonrepelling cycle of period 2. Then since the multiplier of this cycle is

$$\left(Q^2\right)'(\eta - 1) = Q'(\eta - 1)Q'(-\eta - 1) = 1 - 4\eta^2,$$

we get

$$\left|1 - 4\eta^2\right| \leqslant 1. \tag{54}$$

Next we prove if $|1 - 4\eta^2| \leqslant 1$, then Q has two repelling fixed points. In fact, by (54), $\eta^2 + 1 \neq 0$. Thus by (51), we see that Q has two fixed points a and $-a$, where $a \neq 0$ satisfies $a^2 = \eta^2 + 1$. By (51), the multipliers of these two fixed points are

$$\lambda = Q'(\pm a) = 1 \pm 2a \neq 1. \tag{55}$$

As $|1 - 4\eta^2| \leqslant 1$ and $a^2 = \eta^2 + 1$, we have $|4a^2 - 5| \leqslant 1$. Thus we get $|(\lambda - 1)^2 - 5| \leqslant 1$. It follows that either $|(\lambda - 1)^2| > 4$ or $(\lambda - 1)^2 = 4$.

If $(\lambda - 1)^2 = 4$, then $\lambda = -1$ or 3, so that by (55), we see that $a = 1$ and hence $\eta = 0$ by $a^2 = \eta^2 + 1$. A contradiction.

Hence $|(\lambda - 1)^2| > 4$, so that $|\lambda - 1| > 2$, and then $|\lambda| > 1$. Thus the fixed points a and $-a$ are repelling.

Thus in this case

$$\eta \in \left\{z\colon \left|z^2 - \frac{1}{4}\right| \leqslant \frac{1}{4},\ z \neq 0\right\}. \tag{56}$$

Now let $T(z) = \psi^{-1} \circ Q \circ \psi(z)$, where $\psi(z) = z - 1/2$. Then we see that

$$T(z) = z^2 + c \quad \text{and} \quad T^2(z) = z + \left[T(z) - z\right]\left[\left(z + \frac{1}{2}\right)^2 + c + \frac{3}{4}\right], \tag{57}$$

where $c = -\frac{3}{4} - \eta^2$ satisfies

$$|c + 1| \leqslant \frac{1}{4}, \quad c \neq -\frac{3}{4}. \tag{58}$$

Thus P is similar to $z^2 + c$.

Theorem 1 is proved.

5. Proof of Theorem 2

Proof. We consider four cases according to the classification of Theorem 1.

Case 1. Let the polynomial P be in (a). Then

$$P(z) = z^2 + c$$

for some c in the closure $\bar{D}$ of the disk $D = \{z\colon |z+1| < 1/4\}$. We have to show that $\bar{D} \subset \mathcal{M}$. By Lemma 6, it is sufficient to prove $D \subset \mathcal{M}$.

Let $c = -3/4 - \eta^2 \in D$. Then by (57) in the proof of Theorem 1, we see that $P(z) = z^2 + c$ has an attracting periodic cycle $\Gamma = \{\eta - 1/2, -\eta - 1/2\}$ of period 2. By Lemma 3, the immediate basin of attraction associated to Γ contains at least one critical point. As P has only one finite critical point 0, we see that $0 \notin A(\infty)$. By Lemma 5, the Julia set $J(P)$ is connected, and thus $D \subset \mathcal{M}$.

Case 2. Let the polynomial P be in (d). Then

$$P(z) = z^2 - 7/4.$$

By Lemma 6, we have $-7/4 \in \mathcal{M}$. Thus $J(P)$ is connected.

Case 3. Let the polynomial P be in (b). Then

$$P(z) = z^3 - \left(2 + 3c^2\right)z + 2c^3$$

for some c in the closure $\bar{D}$ of the domain $D = \{z\colon |z^2 - 1/36| < 1/36\}$. Let

$$H = \left\{\left(-2 - 3c^2, 2c^3\right)\colon c \in D\right\}.$$

Then we have to show $\bar{H} \subset \mathcal{C}_3$. By Lemma 7, it is sufficient to prove $H \subset \mathcal{C}_3$.

Let $c \in D$. Then by (41)–(43) in the proof of Theorem 1, we see that $P(z)$ has an attracting periodic cycle $\Gamma_1 = \{c+1, c-1\}$ of period 2 and a rationally neutral periodic cycle $\Gamma_2 = \{(-c - \sqrt{9c^2+4})/2, (-c + \sqrt{9c^2+4})/2\}$ of period 2. By Lemma 3, the immediate basins of attraction associated to Γ_1 and Γ_2 contain at least two critical points. Since P has only two finite critical points, we see that there is no finite critical point of P in $A(\infty)$. Thus by Lemma 5, the Julia set $J(P)$ is connected, and thus $H \subset \mathcal{C}_3$.

Case 4. Let the polynomial P be in (c). Then

$$P(z) = z^4 + (-1 \pm 2i)z.$$

By the proof of Theorem 1, we see that $P(z)$ has three rationally neutral periodic cycles Γ_j ($1 \leqslant j \leqslant 3$), of period 2. By Lemma 3, the immediate basins of attraction associated to Γ_j ($1 \leqslant j \leqslant 3$), contain at least three critical points. Since P has only three finite critical points, we see that there is no finite critical point of P in $A(\infty)$. Thus by Lemma 5, the Julia set $J(P)$ is connected.

The proof of Theorem 2 is completed. $\square$

Acknowledgment

The author thanks the referee for his/her helpful and valuable suggestions. In particular, the referee suggested investigating where the exceptional polynomials of degree 2 are located relative the Mandelbrot set.

References

[1] I.N. Baker, Fix-points of polynomials and rational functions, J. London Math. Soc. (2) 39 (1964) 615–622.
[2] W. Bergweiler, Periodische Punkte bei der Iteration ganzer Funktionen, Habilitationsschrift, Rheinisch-Westfälische Techn. Hochsch., Aachen, 1991.
[3] B. Branner, J.H. Hubbard, The iteration of cubic polynomials, Part I: The global topology of parameter space, Acta Math. 12 (1988) 143–206.
[4] L. Carleson, T.W. Gamelin, Complex Dynamics, Springer-Verlag, New York, 1993.
[5] M. Essén, S.J. Wu, Repulsive fixpoints of analytic functions with application to complex dynamics, J. London Math. Soc. (2) 62 (2000) 139–148.